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SOLUBLE SUBGROUPS OF SYMMETRIC AND LINEAR GROUPS

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ABSTRACT
The soluble subgroups of maximal order of the symmetric, alternating, general and special linear groups are determined. Usually, they constitute just one conjugacy class. There are, however, infinitely many exceptions.

In this paper we find the soluble subgroups of maximal order of the symmetric, alternating, general linear and special linear groups (over finite fields). For the linear groups these generally turn out to be the Borel subgroups, i.e., the subgroups that can be written as the set of all triangular matrices, while for the permutation groups, the subgroups we find can be described as the result of packing S_4's, and also S_3's, as densely as possible. One interesting corollary of this determination is that the soluble subgroups of maximal order (of the groups in question) usually constitute just one conjugacy class. However, the linear groups over small fields provide us with infinitely many exceptions to this rule, and also show that the number of classes of our subgroups need not be bounded.

We mention that Y. Segev has carried out a similar investigation for the remaining Chevally groups, again finding that the soluble subgroups of maximal order are the Borel subgroups, excepting some fields [7].

The problem arose originally in connection with a factorization problem. Z. Arad and E. Fisman [1] have determined the simple groups that are products of two subgroups of co-prime order, and have asked about factorizations as products of two soluble subgroups. During the A. Gelbart Symposium at Bar-Ilan University in 1983, D. Gorenstein suggested attacking this problem by considering the maximal order of soluble subgroups of simple groups. Such an application is indeed possible, and will be discussed elsewhere [2]. However, we

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consider the discussion of the soluble subgroups of maximal order of finite groups as a problem of independent interest, and it may be interesting to compare our results to results regarding nilpotent subgroups of maximal order [3, 4]. The maximum order of a soluble permutation group has already been discussed by Dixon [5].

Since we have mentioned general finite groups, let us note the following: Suppose G is a finite group and $N \triangleleft G$. Assume that in N, any class of soluble subgroups of maximal order is also a class of G. Then both the maximal order of a soluble subgroup, and the number of classes of such maximal subgroups, in G, are the products of the corresponding numbers for N and G/N. Our results show that the conditions required from N are always fulfilled if N is a symmetric, alternating, general linear or special linear group, with one single exception. The group SL(2,7) has two classes of soluble subgroups of maximal order, this order being 48. But in the containing group GL(2,7) there is only one such class (the Borel subgroups) and the maximal order is 252.

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1. The symmetric and alternating groups

Let $s(n)$ be the maximal order of a soluble subgroup of S_n, and let L_n denote any soluble subgroup of S_n of order $s(n)$. (Thus L_n is a "generic" notation, in the sense that L_n stands for any one of a given family of subgroups. The notations H_n, K_n, J_n below are used in the same way.)

Write $n = 4k + r$, $0 \leqq r \leqq 3$. Denote $H_n = S_4 \operatorname{wr} L_k \times S_r$. (Again, the notation is "generic"; the exact subgroup H_n depends on the way we divide the set $\{1, \ldots, n\}$ into k quadruples and one r-tuple, and the choice of L_k. We shall not repeat this type of remark for K_n and J_n.)

Let $K_6 = S_3 \operatorname{wr} S_2$, and, for $n \geqq 6$, $K_n = H_{n-6} \times K_6$. Also, $J_9 = S_3 \operatorname{wr} S_3$, and, for $n \geqq 9$, $J_n = H_{n-9} \times J_9$.

THEOREM 1. *The subgroup L_n is of type H_n, with the following exceptions*:
(a) *If* $n \equiv 6 \pmod{16}$, *then L_n is of type K_n.*
(b) *If* $n \equiv 9 \pmod{16}$, *but* $n \not\equiv 25 \pmod{64}$, *then L_n is of type J_n.*

COROLLARY 2. *All soluble subgroups of maximal order of S_n are conjugate.*

This is immediate.

PROOF. We first prove, by induction on n, that $L = L_n$ is of one of the types H_n, K_n or J_n. Later we will sort out the exact type.

Case I. L is intransitive. Suppose L leaves invariant two complementary subsets, of l and m ciphers, where $n = l + m$. Let L induce on these subsets the groups L_l and L_m. By maximality, $L = L_l \times L_m$, and L_l and L_m are indeed (as the notation implies) soluble subgroups of maximal order of S_l and S_m, so we can apply induction to them. Write $l = 4p + s$, $m = 4q + t$, with $0 \leqq s, t \leqq 3$.

Subcase Ia. Both L_l and L_m are of type H. We compare the orders of $L = H_l \times H_m$ and H_n.

$$|H_l \times H_m| = 24^{p+q}s(p)s(q)s!t!, \qquad |H_n| = 24^k s(k)r!$$

Here $k = p + q + 1$ if the pair $\{s,t\}$ is one of $\{1,3\}$, $\{2,2\}$, $\{2,3\}$, $\{3,3\}$, and $k = p + q$ otherwise. Therefore $s(p)s(q) \leqq s(k)$. Using this, and comparing the powers of 24 and $s!t!$ and $r!$, we see that $|L| < |H_n|$ (which is a contradiction), unless $k = p + q$, $s(k) = s(p)s(q)$, and one of s, t is 0, say $s = 0$ and $t = r$. Then $L_p \times L_q$ is an L_k, so

$$L = L_l \times L_m = H_l \times H_m = S_4 \operatorname{wr}(L_p \times L_q) \times S_t = S_4 \operatorname{wr} L_k \times S_r = H_n.$$

Subcase Ib. L_l is of type H and L_m is of type K (or J). Then $L_l \times L_m = H_l \times H_{m-6} \times K_6$. The maximality implies $H_l \times H_{m-6} = L_{l+m-6}$, which by the previous case implies $H_l \times H_{m-6} = H_{n-6}$ and $L_n = K_n$. Similarly, if L_m is of type J, so is L.

Subcase Ic. None of L_l and L_m is of type H. Then $L_l \times L_m$ involves a factor $K_6 \times K_6$, $K_6 \times J_9$ or $J_9 \times J_9$. It is easy to check that these three subgroups, however, have orders less than H_{12}, H_{15} and H_{18}, respectively, so this case cannot occur.

Case II. L is transitive but imprimitive. If L has a system of imprimitivity consisting of m subsets of l elements each, then $n = lm$ and $L = L_l \operatorname{wr} L_m$. Since L is transitive, so are L_l and L_m. Therefore, by the inductive hypothesis, either $4 \mid l$ and $L_l = H_l$, or $L_l = S_2, S_3, K_6$ or J_9. Similarly for L_m.

Subcase IIa. $4 \mid l$ and $L_l = H_l$. Here $H_l = S_4 \operatorname{wr} L_{l/4}$, so

$$L = (S_4 \operatorname{wr} L_{l/4}) \operatorname{wr} L_m = S_4 \operatorname{wr}(L_{l/4} \operatorname{wr} L_m).$$

By maximality, $L_{l/4} \operatorname{wr} L_m = L_{n/4}$, so $L = H_n$.

Subcase IIb. $4 \nmid l$, $4 \mid m$. Here $L_m = S_4 \operatorname{wr} L_{m/4}$ and $L = (L_l \operatorname{wr} S_4) \operatorname{wr} L_{m/4}$, where L_l is S_2, S_3, K_6 or J_9. In all four cases we obtain the contradiction $|L_l \operatorname{wr} S_4| < |H_{4l}|$ $(= |S_4 \operatorname{wr} L_l|)$.

Subcase IIc. $4 \nmid l$, $4 \nmid m$. Here both L_l and L_m are one of S_2, S_3, K_6 and J_9. Here we have $S_3 \operatorname{wr} S_2 = K_6$, $S_3 \operatorname{wr} S_3 = J_9$, and in all other cases $L_l \operatorname{wr} L_m$ is not of maximal order, by

$$|S_2 \operatorname{wr} S_2| < |S_4|, \qquad |S_2 \operatorname{wr} S_3| = 48 < |S_3 \operatorname{wr} S_2| = 72,$$

$$|S_2 \operatorname{wr} K_6| = |S_2 \operatorname{wr} S_3 \operatorname{wr} S_2| < |S_3 \operatorname{wr} S_2 \operatorname{wr} S_2|,$$

$$|S_2 \operatorname{wr} J_9| = |S_2 \operatorname{wr} S_3 \operatorname{wr} S_3| < |S_3 \operatorname{wr} S_2 \operatorname{wr} S_3|,$$

$$|S_3 \operatorname{wr} K_6| = |J_9 \operatorname{wr} S_2| < |H_{18}|,$$

$$|S_3 \operatorname{wr} J_9| = |J_9 \operatorname{wr} S_3| < |H_{27}|,$$

$$|K_6 \operatorname{wr} S_2| = |S_3 \operatorname{wr} S_2 \operatorname{wr} S_2| < |S_3 \operatorname{wr} S_4|,$$

$$|K_6 \operatorname{wr} S_3| = |S_3 \operatorname{wr} S_2 \operatorname{wr} S_3| < |S_3 \operatorname{wr} S_3 \operatorname{wr} S_2|,$$

$$|K_6 \operatorname{wr} K_6| = |S_3 \operatorname{wr} S_2 \operatorname{wr} S_3 \operatorname{wr} S_2| < |S_3 \operatorname{wr} S_3 \operatorname{wr} S_2 \operatorname{wr} S_2|,$$

$$|K_6 \operatorname{wr} J_9| < |S_3 \operatorname{wr} S_3 \operatorname{wr} S_2 \operatorname{wr} S_3|,$$

$$|J_9 \operatorname{wr} K_6| < |H_{27} \operatorname{wr} S_2|, \qquad |J_9 \operatorname{wr} J_9| < |H_{27} \operatorname{wr} S_3|.$$

Case III. L is primitive. Let M be a minimal normal subgroup of L. Then $|M| = n = p^e$, for some prime p, and L contains a subgroup A such that $L = MA$, $M \cap A = 1$, and $C_L(M) = M$ [6, III.3.2]. We can view M as a vector space over GF(p), and then $A \subseteq \mathrm{GL}(e,p)$, so

$$|L| = |M|\,|A| \leqq p^e p^{e^2} = n^{e+1}$$

while

$$s(n) \geqq |H_n| = 24^{[n/4]} r!\, s([n/4]) \geqq 24^{(n-2)/4} s([n/4]) \geqq 24^{(n+2)/4},$$

provided $n \geqq 16$. Here $e = \log_p n \leqq \log_2 n$, and it can be checked that

$$n^{\log_2 n + 1} \leqq 24^{(n+2)/2} \qquad \text{for all } n \geqq 16,$$

so we get a contradiction. Similarly, for all prime powers $n = p^e$ between 4 and 16 we see that $n|\mathrm{GL}(e,p)| < |H_n|$, except for $n = 8$, but in that case we note that A, a proper subgroup of GL(3,2), has order 24 at most, and $|L| \leqq 192 < |H_8|$. Thus L is primitive only for $n \leqq 4$, when $L = S_n = H_n$.

We have by now proved that L_n is indeed one of H_n, J_n, K_n. To complete the proof of Theorem 1, we have to compare the orders of these three subgroups. Writing $n = 4k + r$, these orders are displayed in Table 1.

Using the obvious inequalities $s(k) \geqq s(k-1)$, $s(k) \geqq 2s(k-2)$ we see that $|H_n| > |K_n|$ for $r \neq 2$, $|H_n| > |J_n|$ for $r \neq 1$, $|J_n| = 3|K_n|$ for $r = 1$, and $|K_n| > |J_n|$ for $r = 2$. Thus $L_n = H_n$ for $r = 0,3$, L_n is H_n or J_n for $r = 1$, and is H_n or K_n for $r = 2$.

TABLE 1

r	H_n	K_n	J_n
0	$24^k s(k)$ (for $r = 0, 1$)	$72 \cdot 2 \cdot 24^{k-2} s(k-2) = 6 \cdot 24^{k-1} s(k-2)$	$6^4 \cdot 6 \cdot 24^{k-3} s(k-3) = 324 \cdot 24^{k-2} s(k-3)$
1		$72 \cdot 6 \cdot 24^{k-2} s(k-2) = 18 \cdot 24^{k-1} s(k-2)$	$6^4 \cdot 24^{k-2} s(k-2) = 54 \cdot 24^{k-1} s(k-2)$ (for $r = 1, 2$)
2	$2 \cdot 24^k s(k)$	$72 \cdot 24^{k-1} s(k-1) = 3 \cdot 24^k s(k-1)$ (for $r = 2, 3$)	
3	$6 \cdot 24^k s(k)$		$6^4 \cdot 2 \cdot 24^{k-2} s(k-2) = 108 \cdot 24^{k-1} s(k-2)$

We start with $r = 2$. Then $|H_n|/|K_n| = \frac{2}{3} \cdot s(k)/s(k-1)$ so we want to compare $s(k)/s(k-1)$ with $\frac{3}{2}$. Write $k = 4l + s$, with $0 \leqq s \leqq 3$.

If $s = 0$, then we already know that $L_k = H_k$, $L_{k-1} = H_{k-1}$, and Table 1 shows that

$$\frac{s(k)}{s(k-1)} = 4\frac{s(l)}{s(l-1)} \geqq 4.$$

Thus $L_n = H_n$.

If $s = 1$, then $L_{k-1} = H_{k-1}$ while L_k is H_k or J_k. In the first case $s(k) = s(k-1)$, and in the second

$$\frac{s(k)}{s(k-1)} = \frac{54}{24} \cdot \frac{s(l-2)}{s(l)} \leqq \frac{9}{4} \cdot \frac{1}{2} < \frac{3}{2}$$

so $L_n = K_n$ in both cases.

For $s = 2$, L_{k-1} is H_{k-1} or J_{k-1}, and we have just seen that $|J_{k-1}| \leqq \frac{9}{8}|H_{k-1}|$, so, in any case,

$$\frac{s(k)}{s(k-1)} \geqq \frac{|H_k|}{|L_{k-1}|} \geqq \frac{8}{9}\frac{|H_k|}{|H_{k-1}|} = \frac{16}{9}$$

and $L_n = H_n$.

Finally, for $s = 3$, L_k is H_k, L_{k-1} is H_{k-1} or K_{k-1}, and Table 1 shows that $|K_{k-1}| \leqq \frac{3}{2}|H_{k-1}|$, so

$$\frac{s(k)}{s(k-1)} \geqq \frac{2}{3}\frac{|H_k|}{|H_{k-1}|} = 2.$$

Again $L_n = H_n$, and we see that $L_n = K_n$ precisely for $s = 1$, i.e. $n \equiv 6$ (16).

Now let $r = 1$. We again write $k = 4l + s$, and we are interested in

$$\frac{|H_n|}{|J_n|} = \frac{4}{9} \cdot \frac{s(k)}{s(k-2)} = \frac{4}{9} \cdot \frac{s(k)}{s(k-1)} \cdot \frac{s(k-1)}{s(k-2)}.$$

We have to compare $s(k)/s(k-2)$ with $\frac{9}{4}$.

If $s = 0$ or 1, then from the calculations we made for case $r = 2$, we know that either $s(k)/s(k-1)$ or $s(k-1)/s(k-2) \geqq 4$, while for $s = 3$ we have $s(k)/s(k-1) \geqq 2$ and $s(k-1)/s(k-2) \geqq \frac{16}{9}$. In all these cases $s(k)/s(k-2) > \frac{9}{4}$, so $L_n = J_n$ is possible, if at all, only for $s = 2$, i.e. $n \equiv 9$ (16).

Let $s = 2$. Then L_k is either H_k or K_k, L_{k-1} is either H_{k-1} or J_{k-1}, and $L_{k-2} = H_{k-2}$. If $L_k = H_k$, then $s(k)/s(k-2) = 2$. If $L_k = K_k$, then $k \equiv 6$ (16), so $l \equiv 1$ (4) and

$$\frac{s(k)}{s(k-2)} = \frac{|K_k|}{|H_{k-2}|} = 2\frac{|K_k|}{|H_k|} = 2 \cdot \frac{3}{2} \cdot \frac{s(l-1)}{s(l)}.$$

Here $s(l-1) = |H_{l-1}| = |H_l|$, so $s(l-1)/s(l) = 1$ if $L_l = H_l$, but if $L_l = J_l$, then we already know that $s(l-1)/s(l) = H_l/J_l \geqq \frac{8}{9}$, so $s(k)/s(k-2) \geqq \frac{8}{3} > \frac{9}{4}$ in both cases. Again $L_n = H_n$. Noting that $r = 1$, $s = 2$ and $k \equiv 6$ (16) is the same as $n \equiv 25$ (64), the proof of Theorem 1 is finished.

THEOREM 3. *Let R be a soluble subgroup of maximal order of A_n. Then $R = A_n \cap L$, for some soluble subgroup of maximal order L of S_n.*

PROOF. Note that $|R| \geqq \frac{1}{2}s(n)$. First, assume that R is intransitive, and write $n = l + m$, where R fixes subsets of sizes l and m. Let R_l, R_m be the projections of R on these subsets. If R_l and R_m contain only even permutations, then, by induction, $|R_l| \leqq \frac{1}{2}s(l)$ and $|R_m| \leqq \frac{1}{2}s(m)$ (L_n always contains odd permutations) so $|R| \leqq \frac{1}{4}s(l)s(m) \leqq \frac{1}{4}s(n)$, a contradiction. Thus $R_l \times R_m$ contains odd permutations. But then R is a proper subgroup of this product so

$$|R| \leqq \tfrac{1}{2}|R_l \times R_m| \leqq \tfrac{1}{2}s(l)s(m) \leqq \tfrac{1}{2}s(n)$$

with equality possible only if $R_l \times R_m = L_n$, and $R = L_n \cap A_n$.

Next, if R is transitive but imprimitive, we write $n = lm$, $R \subseteq R_l \operatorname{wr} R_m$, and proceed in exactly the same way as in the intransitive case. So we may assume that R is primitive. As in the proof of Theorem 1 (Case III) we write $R = MA$, with $|M| = n = p^e$ and $|A| \subseteq \mathrm{GL}(e,p)$. We noted, in the proof of Theorem 1, that $|R| \leqq n|\mathrm{GL}(e,p)| < s(n)$, except for $n \leqq 4$ and $n = 8$. If A is a proper subgroup of $\mathrm{GL}(e,p)$, we get actually $|R| < \frac{1}{2}s(n)$, which gives a contradiction also in the present case. Similarly, for $n = 8$ we have $|R| \leqq 192$, $\frac{1}{2}s(8) = 576$. There remains the possibility $A = \mathrm{GL}(e,p)$. This may occur for $e = 2$, $p = 3$, when $n|\mathrm{GL}(2,3)| = 9 \cdot 48 < \frac{1}{2}s(9) = 648$, or for $e = 1$. But for $e = 1$, $n = p$ is prime, and $\mathrm{GL}(1,p)$ contains a cycle of order $p - 1$, which is an odd permutation, so A is again a proper subgroup.

COROLLARY 4. *All soluble subgroups of maximal order of A_n are conjugate.*

2. Linear groups

The natural approach here is, in analogy to the discussion of permutation groups, to distinguish between reducible, imprimitive and primitive groups, and for the last ones to replace Galois' theorem on soluble primitive permutation groups by Suprunenko's detailed description of soluble primitive linear groups [9]. However, Suprunenko's results have been employed already by Wolf [10] and Segev [8] to derive inequalities for the orders of completely reducible soluble linear groups, and we quote rather those results than Suprunenko's, except for small values of n and q, where more care is necessary.

We denote by $T_n = T_n(q)$ the subgroup of $\mathrm{GL}(n, q)$ consisting of all triangular matrices, or any subgroup conjugate to it.

THEOREM 5. *Let S be a soluble subgroup of maximal order of* $\mathrm{GL}(n,q)$ *for* $q \geqq 7$ *or* $q = 4$. *Then S is of type T_n.*

PROOF. First, let S be reducible. Then the elements of S can be put in the form $\left(\begin{smallmatrix} A & 0 \\ C & B \end{smallmatrix}\right)$, where A and B are square matrices of sizes l and m, with $n = l + m$. By induction, we may assume that A and B lie in T_l and T_m, and then $S \subseteq T_n$.

Now let S be irreducible. We aim to derive a contradiction. By Wolf's result referred to above, $|S| < q^{9n/4}$, while $|T_n| = q^{\binom{n}{2}}(q-1)^n$. Since $q - 1 > q^{3/4}$ we get $|T_n| > q^{((n-1)/2+3/4)n}$, and $\frac{1}{2}n + \frac{1}{4} \geqq \frac{9}{4}$ for $n \geqq 4$, so $|S| < |T_n|$. Thus $n = 2$ or 3.

Now if S is imprimitive, then $S = F^* \operatorname{wr} S_n$, where F^* is the multiplicative group of the underlying field, so $|S| = 2(q-1)^2$ for $n = 2$, and $|S| = 6(q-1)^3$ for $n = 3$, but $2(q-1)^2 < q(q-1)^2$ and $6(q-1)^3 < q^3(q-1)^3$, so $|S| < |T_n|$.

Thus S is primitive, and we apply Suprunenko's result, which shows that either $|S| = n(q^n - 1)$ or $|S| = (q-1)n^2|\mathrm{Sp}(2,n)|$. Again, we easily check that

$$2(q^2-1) < q(q-1)^2,\ (q-1)\cdot 2^2 \cdot |\mathrm{Sp}(2,2)| = 24(q-1) < q(q-1)^2 \text{ (for } q \geqq 7),$$
$$3(q^3-1) < q^3(q-1)^3 \quad \text{and} \quad (q-1)\cdot 3^2 \cdot |\mathrm{Sp}(2,3)| = 216(q-1) < q^3(q-1)^3.$$

For $q = 4$, $24(q-1) = 72 > 36 = 4 \cdot 3^2$, but the proof still holds, because actually there is no subgroup of order 72 of $\mathrm{GL}(2,4) \cong Z_3 \times A_5$.

For $q = 2,3,5$ Theorem 5 is false. Thus, $\mathrm{GL}(2,2)$ and $\mathrm{GL}(2,3)$ are themselves soluble, while $\mathrm{GL}(2,5)$ has a subgroup of order $96 > 80 = |T_2(5)|$. For these exceptional values, let us denote by U_2 the subgroup $\mathrm{GL}(2,2)$, $\mathrm{GL}(2,3)$ or the one of order 96 in $\mathrm{GL}(2,5)$ (this is unique up to conjugacy). For $n \geqq 3$, let U_n be the subgroup of block triangular matrices

$$\begin{pmatrix} A_1 & & 0 \\ & \cdots & \\ * & & A_k \end{pmatrix},$$

where each A_i is of size 2×2 if n is even, while one is of size 1×1 and the others 2×2 if n is odd, and $A_i \in U_2$. As usual, a subgroup conjugate to U_n will also be denoted by U_n.

THEOREM 6. *For $q = 2,3,5$, each soluble subgroup S of maximal order of* $\mathrm{GL}(n,q)$ *is of type U_n.*

COROLLARY 7. *If either $q \neq 2,3,5$ or n is even,* $\mathrm{GL}(n,q)$ *has just one class of soluble subgroups of maximal order. For $q = 2,3,5$ and n odd, there are $\frac{1}{2}(n+1)$ such classes.*

The first part of the Corollary is obvious. The second follows from the fact that U_n's, in which the 1×1 block is not in the same position, cannot be conjugate, because U_n determines its unique chain of invariant subspaces of the n-dimensional space on which the matrices act.

PROOF OF THEOREM 6. This is about the same as the one for Theorem 5. In the same way we see that we may take S to be irreducible, and then that $n = 2$ or 3. For $n = 2$, we have chosen U_2 to be the unique (up to conjugacy) soluble subgroup of maximal order, so let $n = 3$. Then $6(q-1)^3 < |T_3| < |U_3|$, so S is primitive and has order $3(q^3-1)$ or $216(q-1)$, while $|U_3(2)| = 24$, $|U_3(3)| = 864$, $|U_3(5)| = 9600$. The possibility $|S| = 216(q-1)$ certainly does not occur for $q = 2$, where $|\mathrm{GL}(3,2)| = 168$, and so $|S| < |U_3(q)|$ in all cases.

THEOREM 8. *Let S be a soluble subgroup of maximal order of* $\mathrm{SL}(n,q)$. *Except for $q = 7$, $n = 2$, there exists a soluble subgroup of maximal order in* $\mathrm{GL}(n,q)$, *say S_1, such that $S = S_1 \cap \mathrm{SL}(n,q)$. The number of classes of such subgroups in* $\mathrm{SL}(n,q)$ *is the same as in* $\mathrm{GL}(n,q)$.

PROOF. First, let S be reducible, so its elements have the form

$$X = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix},$$

with A and B of sizes l and m, and $l + m = n$. Suppose that in the compound homomorphism $X \to A \to \det A$, S is mapped into a subgroup of order d of F^*. Then in the homomorphism $X \to B \to \det B$, S is mapped onto the same subgroup. Let V_n denote a soluble subgroup of maximal order of $\mathrm{GL}(n,q)$ (i.e. T_n or U_n), and note that

$$|\mathrm{SL}(n,q) \cap V_n| = \frac{1}{q-1} |V_n|.$$

If S maps onto the subgroups S_l and S_m of $\mathrm{GL}(l,q)$ and $\mathrm{GL}(m,q)$ (by $X \to A$ and $X \to B$), then $|S_l : S_l \cap \mathrm{SL}(l,q)| = d$, and by induction

$$|S_l| \leqq \frac{d}{q-1}|V_l|,$$

and similarly

$$|S_m| \leqq \frac{d}{q-1}|V_m|.$$

(Here we assume $q \neq 7$; the case $q = 7$ will be discussed separately.) Let W be the subgroup of $\mathrm{GL}(n,q)$ consisting of all matrices of block form as X above, with $A \in S_l$ and $B \in S_m$. Then $S \subseteq W$, $S = W \cap \mathrm{SL}(n,q)$ (by maximality of S), $|W:S| = d$ and

$$|W| = q^{lm}|S_l||S_m| \leqq \frac{d^2}{(q-1)^2}q^{lm}|V_l||V_m| \leqq \frac{d^2}{(q-1)^2}|V_n|,$$

$$|S| = \frac{1}{d}|W| \leqq \frac{d}{q-1}\cdot\frac{1}{q-1}|V_n| \leqq |V_n \cap \mathrm{SL}(n,q)|.$$

Thus $(1/(q-1))|V_n|$ is the maximal possible order for S, and we see that it is realized only when $d = q-1$ and $W = V_n$, so $S = V_n \cap \mathrm{SL}(n,q)$.

Now let S be irreducible, so $|S| < q^{9n/4}$. For $n \geqq 5$, and $q \geqq 4$, we get

$$|T_n \cap \mathrm{SL}(n,q)| = q^{\binom{n}{2}}(q-1)^{n-1} > q^{\binom{n}{2}+3(n-1)/4} > q^{9n/4},$$

while for $q = 2,3$ we have

$$|U_n \cap \mathrm{SL}(n,q)| = q^{\binom{n}{2}}(q-1)^{n-1}(q+1)^{[\frac{n}{2}]} \geqq q^{\binom{n}{2}+(n-1)/2} > q^{(n^2-1)/2} > q^{9n/4}.$$

Thus $n \leqq 4$ for all q.

If $S \subseteq F^* \operatorname{wr} S_n$, then

$$|S| \leqq \frac{1}{(q-1)}(q-1)^n n!,$$

since $F^* \operatorname{wr} S_n$ contains elements of all possible determinants, and we have already checked that

$$|S| \leqq \frac{1}{q-1}|T_n| \qquad \text{for } n = 2,3,$$

and this holds also for $n = 4$ ($24(q-1)^3 < q^6(q-1)^3$). If S is imprimitive, it is still possible that $n = 4$, and $S \subseteq T \operatorname{wr} S_2$, for some $T \subseteq \mathrm{GL}(2,q)$. Then

$$|S| \leqq 2(q^2-1)^2(q^2-q)^2 = 2q^2(q-1)^4(q+1)^2 < q^6(q-1)^3 = |T_4 \cap \mathrm{SL}(4,q)|$$

for $q \geqq 3$. The case $q = 2$ needs no treatment, as $\mathrm{SL}(n,2) = \mathrm{GL}(n,2)$.

So, S is primitive. Then so is $R = SZ(\mathrm{GL}(n,q))$, of order $|S|(q-1)/(n,q-1)$ and we apply Suprunenko's results. We have

$$|S| \leqq \frac{n}{q-1}|R|,$$

and we want

$$|S| < \frac{1}{q-1}|V_n|,$$

so it suffices to show that $n|R| < |V_n|$. The possible values for $|R|$ were listed in the proof of Theorem 5, and we have to add to them, for $n = 4$, the possibility $|R| = 2(q^2-1)\cdot 2^2\cdot|\mathrm{Sp}(2,2)|$. Also, for $n = 4$ we have $|\mathrm{Sp}(4,2)|$ rather than $|\mathrm{Sp}(2,4)|$, and this can be replaced by 72, the maximal order of a soluble subgroup of $\mathrm{Sp}(4,2) \cong S_6$. We check the inequalities for each dimension.

For $n = 2$, $4(q^2-1) < q(q-1)^2$ holds for $q \geqq 7$, and

$$2(q-1)\cdot 2^2|\mathrm{Sp}(2,2)| = 48(q-1) < q(q-1)^2 \qquad \text{for } q \geqq 8.$$

The values $q = 4,5$ are checked individually, using $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5) \cong A_5$, to show that $S = \mathrm{SL}(2,q) \cap V_2$. Similarly for $q = 3$, where $V_2 = \mathrm{GL}(2,3)$.

For $n = 3$, $9(q^3-1) < q^3(q-1)^3$ holds for $q \geqq 4$, and

$$3(q-1)\cdot 3^2\cdot|\mathrm{Sp}(2,3)| = 648(q-1) < q^3(q-1)^3 \qquad \text{for } q \geqq 5.$$

For $q = 3$ we can replace $q^3(q-1)^3$ by $|V_3| = 864$, and the left hand sides by $(n,q-1)|R| = |R|$, so the inequalities still hold. For $q = 4$, the second inequality is violated. However, this inequality corresponds to the case

$$|R| = (q-1)\cdot 3^2\cdot|\mathrm{Sp}(2,3)| = 216(q-1) = 648,$$

and $|S| \geqq |T_3 \cap \mathrm{SL}(n,q)| = q^3(q-1)^2 = 576$ holds only if $|S| = |R|$, which is impossible because 648 does not divide $|\mathrm{SL}(3,4)| = 60480$.

Finally, for $n = 4$, we want $16(q^4-1)$, $192(q^2-1)$ and $4608(q-1)$ to be less than $q^6(q-1)^4$, and this is indeed true for all $q \geqq 3$.

Next, let S and T be two soluble subgroups of maximum order of $\mathrm{SL}(n,q)$, and write them as $S = S_1 \cap \mathrm{SL}(n,q)$, $T = T_1 \cap \mathrm{SL}(n,q)$. Then

$$|S_1 : S| = |T_1 : T| = |\mathrm{GL}(n,q) : \mathrm{SL}(n,q)|,$$

$S \Delta S_1$, $T \Delta T_1$ and S and T, as maximal soluble subgroups, are self-normalizing in $\mathrm{SL}(n,q)$. Therefore S_1 and T_1 are the normalizers of S and T in $\mathrm{GL}(n,q)$. It follows that S and T are conjugate in $\mathrm{SL}(n,q)$ if and only if S_1 and T_1 are conjugate in $\mathrm{GL}(n,q)$.

The case $q = 7$ is truly exceptional in Theorem 8. Thus SL(2,7), of order 336, has two classes of soluble subgroups of maximal order 48, while GL(2,7) has only one such class of order 252, which intersects SL(2,7) in subgroups of order 42. These (well-known) facts follow from Theorems 5 and 6, remembering that $\mathrm{PSL}(2,7) \cong \mathrm{GL}(3,2)$, the simple group of order 168.

To deal with the case $q = 7$, $n > 2$ we note first that the two classes of subgroups of order 48 become one class in GL(2,7), and this means that the largest subgroup of GL(2,7) containing such a subgroup of order 48, which is its normalizer, has order $3 \cdot 48 = 144$. The proof of Theorem 8 needs modification of only one point to accomodate this case, namely, when invoking induction for a reducible S. There, if $l = 2$, we have to write $|S_l| \leqq d \cdot 48$ rather than

$$|S_l| \leqq \frac{d}{q-1} |V_2| \quad (= d \cdot 42),$$

but only if $d \leqq 3$. If also $m = 2$, we check that

$$|S| \leqq d \cdot 48^2 \cdot 7^4 \leqq 3 \cdot 48^2 \cdot 7^4 < 7^6 \cdot 6^3 = |T_4 \cap \mathrm{SL}(4,7)|,$$

and, for any m, we check that

$$|S| \leqq \frac{1}{d} \cdot d \cdot 48 \cdot \frac{d}{6} \cdot 7^{\binom{m}{2}} \cdot 6^m < \frac{1}{6} \cdot 7^{\binom{n}{2}} \cdot 6^n,$$

with $n = m + 2$ and $d \leqq 3$. This ends the proof.

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